PHYS4150 — PLASMA PHYSICS

LECTURE 5 - SINGLE PARTICLE MOTION IN AN UNIFORM B FIELD

Sascha Kempf[*]

G135, University of Colorado, Boulder

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Single particle motion in an uniform B field

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1 EXAMPLE: SPONTANEOUS CHARGE FLUCTUATIONS

Let's calculate the radius r_m of a sphere that could be spontaneously depleted of all electrons due to thermal fluctuations. In this case all electrons previously within the sphere

$$N_{e^-} = \frac{4}{3}\pi r_m^3 n$$

would move through the outer boundary of the sphere, resulting in a total ion charge within the sphere of $Q = eN_{e^-}$. The corresponding electric field E_r can be found from Gauss' law

$$Q = \epsilon_0 \oint \mathbf{E}\,\mathrm{d}\mathbf{A} = \epsilon_0 E_r \oint \mathrm{d}\mathbf{A} = \epsilon_0 E_r 4\pi r_m^2,$$

$$E_r = \frac{Q}{4\pi\epsilon_0 r_m^2}.$$

Because the energy density of an electric field is $\frac{1}{2}\epsilon_0 E^2$, we can can calculate the total energy resulting from breaking the charge neutrality:

$$W = \int_0^{r_m} \frac{1}{2}\epsilon_0 E^2 4\pi\epsilon_0 r^2\,\mathrm{d}r = \pi r_m^5 \frac{2n^2e^2}{45\epsilon_0}.$$

[*] sascha.kempf@colorado.edu

We wish W to be equal to thermal energy of the electrons

$$W_{th} = \frac{3}{2} n k_B T \frac{4}{3} \pi r_m^3 = W,$$

which implies that

$$r_m^2 = 45 \frac{\epsilon_0 k_B T}{ne^2} = 45 \lambda_D^2,$$

or

$$\boxed{r_m \approx 7\lambda_D}$$

2 SINGLE PARTICLE MOTION IN A UNIFORM B FIELD

Before we deal with the really messy stuff, it is beneficial to study the motion of single charged particles in uniform electric and magnetic fields. As a first step let's investigate the case of a charged grain moving in an uniform magnetic field.

You will show in your homework assignment that if only a Lorentz force

$$\mathbf{F}_L = q(\mathbf{v} \times \mathbf{B})$$

acts on a charged particle, its kinetic energy $T = \frac{1}{2} m \mathbf{v}^2$ is an integral of motion. We now split $\mathbf{v}$ into its components parallel and orthogonal to the magnetic field:

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

and similarly

$$T = \frac{1}{2} m \mathbf{v}_{\parallel}^2 + \frac{1}{2} m \mathbf{v}_{\perp}^2 = T_{\parallel} + T_{\perp}.$$

Because of $\mathbf{v} \times \mathbf{B} = \mathbf{v}_{\parallel} \times \mathbf{B} + \cancel{\mathbf{v}_{\perp} \times \mathbf{B}} = \mathbf{v}_{\parallel} \times \mathbf{B}$, $T_{\parallel}$ is an integral of motion, and $T_{\perp}$ is an integral of motion, too.

Then, the equation of motion for the component $\mathbf{v}_{\perp}$

$$m \frac{d\mathbf{v}_{\perp}}{dt} = q \mathbf{v}_{\perp} \mathbf{B} = m \frac{\mathbf{v}_{\perp}^2}{\rho_c}$$

describes a circular motion around the so-called guiding center. The radius

$$\boxed{\rho_c = \frac{m v_{\perp}}{|q| B}} \tag{1}$$

is the *cyclotron* or *Lamor radius*. The angular frequency of the cyclotron motion

$$\boxed{\omega_c = \frac{v_{\perp}}{\rho_c} = \frac{|q| B}{m}} \tag{2}$$

is called the *cyclotron* or *Lamor frequency*.

Note that the gyromotion of a charge constitutes a current loop

$$j = \frac{q}{\Delta t} = q\frac{\omega_c}{2\pi} = \frac{q}{2\pi}\frac{qB}{m} = \frac{q^2}{2\pi}\frac{B}{m}.$$

In this case, there is a *magnetic moment*

$$\mu = \text{area} \cdot \text{current}.$$

The area enclosed by the loop current is

$$A_{loop} = \pi\rho_c^2 = \pi\frac{m^2 v_\perp^2}{|q|^2 B^2},$$

and thus

$$\mu = j \cdot A_{loop} = \frac{q^2}{2\pi}\frac{B}{m}\pi\frac{m^2 v_\perp^2}{|q|^2 B^2} = \frac{m v_\perp^2}{2B},$$

and finally

$$\boxed{\mu = \frac{T_\perp}{B}}. \tag{3}$$

Note that for both, the electrons and the ions, the direction of μ is opposite to the applied magnetic field **B**. This means that μ resulting from the plasma particles' gyromotion weakens the applied field – the plasma is *diamagnetic*.